JOURNAL OF APPROXIMATION THEORY 55, 65–85 (1988)

Sequential Convergence and Approximation in the Space of Riemann Integrable Functions*

W. DICKMEIS, H. MEVISSEN, R. J. NESSEL, AND E. VAN WICKEREN

Lehrstuhl A für Mathematik, Rheinisch-Westfälische Technische Hochschule Aachen, D-5100 Aachen, West Germany

Communicated by P. L. Butzer

Received July 8, 1986

1. INTRODUCTION

The purpose of this paper is to introduce a concept of sequential convergence in the space of Riemann integrable functions (in the classical sense) and to discuss some consequences in approximation theory. In fact, our approach originates in work of Pólya [12] on the convergence of quadrature formulas.

In [12] Pólya first treated necessary and sufficient conditions for convergence of quadrature formulas for continuous functions. From an abstract point of view this is covered by the theorem of Banach–Steinhaus. Pólya then extended the results to Riemann integrable functions. Though well known, too, this part of his treatment remained somewhat isolated, at least to our knowledge. Among other things it follows, however, that if one introduces an appropriate concept of convergence, then also this part of Pólya's work may be reestablished by an application of a theorem of Banach–Steinhaus-type.

The notion of convergence in question (for bounded functions of several variables) is given in Definition 2.1. It not only turns out that the space of Riemann integrable functions is (sequentially) complete, but continuous functions are dense in it. The latter fact enables one to discuss approximation.

In this connection our first topic is concerned with theorems of Banach–Steinhaus-type. Here the situation is rather clarified for (sublinear) functionals. To this end, Section 3 first relates various concepts of continuity. Then Theorem 4.3 states that a sequence of sublinear, Riemann continuous functionals converges for each Riemann integrable function if and only if it converges for each element of a Riemann dense subset and if

* Research (partially) supported by Deutsche Forschungsgemeinschaft Grant No. Ne 171/6.

0021-9045/88 $3.00

the sequence is equi-Riemann-continuous or continuously Riemann convergent. Since, however, point evaluation functionals cannot be Riemann continuous, this does not yet cover Pólya's result. But it can be shown (cf. Theorem 4.4) that continuous Riemann convergence, which in fact reflects the original Pólya condition on the semicontinuity of some "Intervallvereinfunktion," is the appropriate notion to formulate an equivalence assertion even for quadrature (cubature) formulas.

Section 5 considers the extension of the classical approximate identity argument (peaking property) in order to derive a sufficient criterion for the Riemann convergence of a sequence of operators, mapping the space of Riemann integrable functions into itself. Since Theorem 5.1 does not assume any Riemann continuity, applications are even possible if point evaluation functionals are involved, thus, e.g., to Bernstein polynomials (cf. Corollary 5.3).

Along the same lines many other topics of approximation theory may now be extended to Riemann integrable functions. See [10a-c, 15–17] for some further material.

2. Riemann Convergence

Let $\mathbb{N}$, $\mathbb{P}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of natural, non-negative integral, real, and complex numbers, respectively, and let $\mathbb{R}^N$, $N \in \mathbb{N}$, be the Euclidean N-space with $|x|^2 := \sum_{j=1}^N x_j^2$. In the following consider a (fixed, non-trivial) compact interval $[a, b]$ where $a, b \in \mathbb{R}^N$ are such that $a_j < b_j$. Let V denote the family of finite unions of (not necessarily closed or disjoint) subintervals, also called "Intervallvereine" in [12] or elementary sets in [2, p. 252]. Thus V is an algebra, i.e., $I \cap J$, $I \cup J$, and the complement $\complement I$ belong to V if $I, J \in V$.

Concerning integrals, the upper and lower Riemann integrals of $f \in B = B[a, b]$, the space of functions, everywhere defined and bounded on $[a, b]$ with norm $\|f\| = \|f\|_B := \sup\{|f(x)| : x \in [a, b]\}$, are denoted by

$$\overline{\int} f := \overline{\int}_{[a,b]} f(x)\,dx, \qquad \underline{\int} f := \underline{\int}_{[a,b]} f(x)\,dx,$$

respectively. By Riemann's criterion $f \in B$ is Riemann integrable on $[a, b]$, i.e., $f \in R = R[a, b]$, if and only if $\overline{\int} f = \underline{\int} f$, in which case one has for the Riemann integral,

$$\int f := \int_{[a,b]} f(x)\,dx = \overline{\int} f \left(= \underline{\int} f \right).$$

For arbitrary $A \subset [a, b]$ let χ_A be the characteristic function, i.e., $\chi_A(x) = 1$ if $x \in A$ and $=0$ otherwise. The outer and inner Riemann measure (Jordan content) may then be defined by

$$\bar{\mu}(A) := \overline{\int} \chi_A = \inf\{\mu(I) : I \in V, I \supset A\},$$

$$\underline{\mu}(A) := \underline{\int} \chi_A = \sup\{\mu(I) : I \in V, I \subset A\},$$

respectively, where $\mu(I) := \int \chi_I$ is the elementary content. Note that A is Riemann measurable if and only if $\bar{\mu}(A) = \underline{\mu}(A)$. In this case $\chi_A \in R$ and the Riemann measure of A is given by $\mu(A) = \int \chi_A$.

Clearly, $C \subset R \subset B$ where $C = C[a, b]$ is the space of continuous functions on $[a, b]$. In fact, each of these spaces, thus in particular R, as endowed with $\|\cdot\|$, is a Banach space. But this norm is not appropriate for approximation in R since, e.g., C is not dense in R. This disadvantage is avoided by the following concept of sequential convergence, in fact even well-defined on B.

Definition 2.1. *A sequence $\{f_n\} \subset B$ is called (Riemann) R-convergent to $f \in B$ (in notation, R-$\lim f_n = f$) if, for $n \to \infty$,*

$$\|f_n\| = \mathcal{O}(1), \tag{2.1}$$

$$\overline{\int} \sup_{k \geq n} |f_k - f| = o(1). \tag{2.2}$$

Obviously, (2.1) ensures the existence of the integral in (2.2). Moreover, R-convergence is linear since $\overline{\int} |f|$ is a seminorm on B. In the following, $I_n \downarrow 0$ denotes a sequence $I_n \in V$ with $I_{n+1} \subset I_n$ and $\mu(I_n) = o(1)$.

Proposition 2.2. *Let $\{f_n\} \subset B$ satisfy (2.1) and let $f \in B$. The following assertions are equivalent ($n \to \infty$):*

(i) *R-$\lim_{n \to \infty} f_n = f$.*

(ii) *For each $\varepsilon > 0$,*

$$\bar{\mu}(\{x \in [a, b] : \sup_{k \geq n} |f_k(x) - f(x)| \geq \varepsilon\}) = o(1). \tag{2.3}$$

(iii) *Lebesgue almost everywhere on $[a, b]$,*

$$\sigma(\sup_{k \geq n} |f_k - f|, x) = o(1), \tag{2.4}$$

where σ *is defined for* $g \in B$ *and* $x \in [a, b]$ *by*

$$\sigma(g, x) := \inf_{\delta > 0} \sup_{\substack{y \in [a,b] \\ |y-x| < \delta}} |g(y)|. \tag{2.5}$$

(iv) *For each* $\varepsilon > 0$ *there exists* $I_n \downarrow 0$ *such that*

$$|f_n(x) - f(x)| < \varepsilon \qquad (x \in \complement\, I_n).$$

Proof. For abbreviation set

$$h_n := \sup_{k \geqslant n} |f_k - f|, \qquad B_n^\varepsilon := \{x \in [a, b] : h_n(x) \geqslant \varepsilon\}.$$

Then (i) implies (ii) since for $\varepsilon > 0$,

$$\bar{\mu}(B_n^\varepsilon) := \overline{\int} \chi_{B_n^\varepsilon} \leqslant \frac{1}{\varepsilon} \overline{\int} h_n \chi_{B_n^\varepsilon} \leqslant \frac{1}{\varepsilon} \overline{\int} h_n = o(1).$$

On the other hand, if (ii) is fulfilled, then for each $\varepsilon > 0$,

$$\overline{\int} h_n \leqslant \overline{\int} h_n \chi_{B_n^\varepsilon} + \overline{\int} h_n (1 - \chi_{B_n^\varepsilon}) \leqslant \|h_n\| \; \bar{\mu}(B_n^\varepsilon) + \varepsilon\mu([a, b])$$

so that (i) follows in view of (2.1), (2.3).

Concerning (ii) $\Leftrightarrow$ (iii), let us first show that ($\bar{A}$ denotes the closure of A)

$$\overline{B_n^\varepsilon} \subset C_n^\varepsilon := \{x \in [a, b] : \sigma(h_n, x) \geqslant \varepsilon\} \subset \overline{B_n^{\varepsilon/2}}. \tag{2.6}$$

Indeed, if $x \in \overline{B_n^\varepsilon}$, then for each $\delta > 0$ there exists $z \in B_n^\varepsilon$ with $|z - x| < \delta$. Thus $x \in C_n^\varepsilon$ since

$$\sup_{|y-x|<\delta} h_n(y) \geqslant h_n(z) \geqslant \varepsilon.$$

Moreover, $x \in C_n^\varepsilon$ implies that for any $\delta > 0$ there is $y \in B_n^{\varepsilon/2}$ with $|y - x| < \delta$, thus $x \in \overline{B_n^{\varepsilon/2}}$. Let ∂A denote the boundary of $A \subset [a, b]$. Then $\bar{\mu}(\partial A) = \bar{\mu}(A) - \underline{\mu}(A)$ (cf. [2, p. 256]) so that $\bar{\mu}(A) \leqslant \bar{\mu}(\bar{A}) \leqslant 2\bar{\mu}(A)$. Since $\bar{A}$ is compact, one also has $\bar{\mu}(\bar{A}) = \bar{\lambda}(\bar{A})$ with outer Lebesgue measure $\bar{\lambda}$ (cf. [2, p. 259]). In view of (2.6) this implies that (2.3) is equivalent to $\bar{\lambda}(C_n^\varepsilon) = o(1)$ for each $\varepsilon > 0$, thus if and only if

$$A := \{x \in [a, b] : \inf_{n \in \mathbb{N}} \sigma(h_n, x) > 0\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} C_n^{1/m}$$

is of Lebesgue measure zero, hence (iii) since $\sigma(h_n, x)$ decreases.

If (iv) is satisfied, then $B_n^\varepsilon \subset I_n$ since $\complement I_n \subset \complement I_k$ for $k \geqslant n$, thus (ii) follows in view of $\mu(I_n) = o(1)$. On the other hand, if $\bar{\mu}(B_n^\varepsilon) = o(1)$, there exists $J_n \in V$ with $B_n^\varepsilon \subset J_n$ and $\mu(J_n) \leqslant \bar{\mu}(B_n^\varepsilon) + 1/n$. Setting $I_n := \bigcap_{k=1}^n J_k \in V$, one has $I_{n+1} = I_n \cap J_{n+1}$ so that $I_n \downarrow 0$ and

$$B_n^\varepsilon = \bigcap_{k=1}^{n} B_k^\varepsilon \subset \bigcap_{k=1}^{n} J_k = I_n.$$

This yields (iv). ∎

At this point let us compare the Riemann convergence with the usual ones of Lebesgue theory. First of all, the use of the monotone quantity $\sup_{k \geqslant n} |f_k - f|$ compensates the missing countable additivity of the algebra of Riemann measurable sets. In many proofs the equiboundedness (2.1) serves as a substitute for arguments, based on Lebesgue dominated convergence. Obviously, uniform convergence, i.e., convergence in B-norm, implies R-convergence. On the other hand, R-convergence as induced on R in particular implies $\int |f_n - f| = o(1)$, thus convergence in $L^1(a, b)$-norm. In fact, (2.1), (2.2) strengthen L^1-convergence on R to ensure that limits continue to stay in R (cf. Theorem 2.5). Moreover, pointwise convergence Lebesgue almost everywhere is a consequence of (iii), thus of R-convergence. Assertion (ii) may be compared with convergence in measure and (iv) with almost uniform convergence.

Of course, the structure of the two conditions (2.1), (2.2) is also connected with abstract concepts like those of two-norm convergence or Saks spaces (cf. [1, 11]).

Note that Definition (2.5) is closely related to that of the oscillation

$$o(g, x) := \inf_{\delta > 0} \sup\{|g(y) - g(z)| : y, z \in [a, b], |y - x| < \delta, |z - x| < \delta\}. \tag{2.7}$$

In this connection Conditions (ii) and (iii) look rather similar to the two versions of Lebesgue's theorem concerning Riemann integrability (cf. [2, pp. 230, 259]).

As in Lebesgue theory the limit of an R-convergent sequence is not unique so that the following gives rise to the introduction of the equivalence classes

$$[f] := \left\{ g \in B : \bar{\int} |f - g| = 0 \right\}. \tag{2.8}$$

LEMMA 2.3. *Let $f, g, f_n \in B$ with R-$\lim f_n = f$. Then R-$\lim f_n = g$ if and only if $g \in [f]$.*

Proof. The equivalence immediately follows by the inequalities

$$\overline{\int} |f-g| \leqslant \overline{\int} \sup_{k \geqslant n} |f-f_k| + \overline{\int} \sup_{k \geqslant n} |g-f_k|,$$

$$\overline{\int} \sup_{k \geqslant n} |f_k-g| \leqslant \overline{\int} \sup_{k \geqslant n} |f_k-f| + \overline{\int} |f-g|. \quad \blacksquare$$

Obviously, each R-convergent sequence $\{f_n\} \subset B$ is an R-Cauchy sequence, i.e., satisfies (2.1) and

$$\lim_{n \to \infty} \overline{\int} \sup_{j,k \geqslant n} |f_j - f_k| = 0. \tag{2.9}$$

In fact, the converse is valid as well.

PROPOSITION 2.4. *B is (sequentially) R-complete.*

Proof. Let $\{f_n\} \subset B$ satisfy (2.1), (2.9). Setting

$$f(x) := \limsup_{n \to \infty} f_n(x), \tag{2.10}$$

it follows by (2.1) that $f \in B$ with

$$|f_k(x) - f(x)| \leqslant \sup_{j \geqslant n} |f_k(x) - f_j(x)| \qquad (n \in \mathbb{N}).$$

In view of (2.9) this yields $(n \to \infty)$

$$\overline{\int} \sup_{k \geqslant n} |f_k - f| \leqslant \overline{\int} \sup_{j,k \geqslant n} |f_k - f_j| = o(1),$$

thus $R\text{-}\lim f_n = f$. $\blacksquare$

As already mentioned, it is useful to have R-convergence well-defined on the whole set B, but R-convergence turns out to be particularly significant when considered on the subset R.

THEOREM 2.5. *R is (sequentially) R-complete. Moreover, $[f] \subset R$ if and only if $f \in R$.*

Proof. Let $\{f_n\} \subset R$ be an R-Cauchy sequence. By Proposition 2.4 $\{f_n\}$ is R-convergent to some $f \in B$. To show that even $f \in R$, by Riemann's criterion (cf. [2, p. 254])

$$0 \leqslant \left(\overline{\int} - \underline{\int}\right) f = \left(\overline{\int} - \underline{\int}\right)(f - f_n) \leqslant 2 \overline{\int} |f_n - f| = o(1),$$

thus $f \in R$. Moreover, if $g \in [f]$ with $f \in R$, then

$$0 \leqslant \left(\overline{\int} - \underline{\int}\right)(f-g) \leqslant 2 \overline{\int} |f-g| = 0,$$

thus $f - g \in R$, and therefore $g \in R$, too. ▌

From the point of view of approximation, the most important feature of R-convergence is the fact that the R-closure of standard classes of smooth functions yields R (and not B). In fact,

THEOREM 2.6. *C is R-dense in R.*

Proof. Obviously, one may restrict oneself to the unit cube $[a, b] = [0, \bar{1}]$, $\bar{1} := (1, ..., 1) \in \mathbb{R}^N$. For one-dimensional $t \in [0, 1]$ define

$$g_{jn}(t) := \begin{cases} 1 - |j - nt|, & |j - nt| \leqslant 1 \\ 0, & \text{else} \end{cases} \tag{2.11}$$

so that g_{jn} is continuous with

$$\sum_{j=0}^{n} g_{jn}(t) = 1 \qquad (t \in [0, 1]). \tag{2.12}$$

For a multiindex $i \in \Gamma_n := \{(i_1, ..., i_N) \in \mathbb{P}^N : i_j \leqslant n\}$ set $(x \in [0, \bar{1}],\ f \in R)$

$$G_{in}(x) := \prod_{j=1}^{N} g_{i_j n}(x_j), \qquad T_n f(x) := \sum_{i \in \Gamma_n} f(i/n)\, G_{in}(x).$$

Then T_n is a positive, linear operator with (cf. (2.12))

$$T_n 1 = \sum_{i \in \Gamma_n} G_{in} = \sum_{i_1=0}^{n} g_{i_1 n} \cdots \sum_{i_N=0}^{n} g_{i_N n} = 1. \tag{2.13}$$

Now let $f \in R$. To show that the continuous functions $g_n := T_{2^n} f$ are R-convergent to f, first of all note that $\|T_n f\| \leqslant \|f\|$ (cf. (2.13)), thus (2.1). In view of Riemann's criterion,

$$\sum_{i \in \Gamma_{n-1}} (M_{in} f - m_{in} f)\, n^{-N} =: \delta_n = o(1),$$

$$M_{in} f := \sup_{x \in S_{in}} f(x), \qquad m_{in} f := \inf_{x \in S_{in}} f(x),$$

$S_{in} := [i/n,\ (i + \bar{1})/n]$ denoting the subintervals of the given equidistant partition. For $\varepsilon > 0$ and

$$A_n := \bigcup \{S_{in} : M_{in} f - m_{in} f \geqslant \varepsilon/2\}$$

one therefore has that

$$\bar{\mu}(A_n) = \sum_{M_{in}f - m_{in}f \geqslant \varepsilon/2} n^{-N} \leqslant \frac{2}{\varepsilon} \sum_{i \in \Gamma_{n-1}} (M_{in}f - m_{in}f)\, n^{-N} = 2\delta_n/\varepsilon = o(1). \tag{2.14}$$

On the other hand, if $x \in \complement A_n$, there exists $i_0 \in \Gamma_{n-1}$ such that $x \in S_{i_0 n}$ and $M_{i_0 n}f - m_{i_0 n}f < \varepsilon/2$. Then $G_{in}(x) \neq 0$ only if $i/n \in S_{i_0 n}$. Therefore (cf. (2.13))

$$|T_n f(x) - f(x)| \leqslant \sum_{i/n \in S_{i_0 n}} |f(i/n) - f(x)|\, G_{in}(x)$$
$$\leqslant \sum_{i/n \in S_{i_0 n}} (M_{i_0 n}f - m_{i_0 n}f)\, G_{in}(x) < \varepsilon/2.$$

Now consider the subsequences T_{2^n} and $A_{2^n} \in V$. Since $\{S_{i,2^n}: i \in \Gamma_{2^n-1}\}$ corresponds to a refining family of partitions, $A_{2^n} \downarrow 0$ by (2.14) with

$$|T_{2^n} f(x) - f(x)| \leqslant \varepsilon/2 < \varepsilon \qquad (x \in \complement A_{2^n}).$$

Hence condition (iv) of Proposition 2.2 is satisfied and the assertion is shown. ∎

Let us emphasize that the continuous functions $g_n = T_{2^n} f$, approximating $f \in R$, are defined via the multivariate spline operator T_n, the knots of which are independent of f. So this constructive proof yields a first approximation process. Moreover, with a slight modification to g_{jn} in (2.11), for example, $g_{jn} := \chi_{[j/n,(j+1)/n)}$, the above proof delivers

COROLLARY 2.7. *The set of step functions*

$$g = \sum_{\text{finite}} \alpha_j \chi_{I_j} \qquad (\alpha_j \in \mathbb{C}, I_j \in V)$$

is dense in R.

Summarizing, Theorems 2.5, 2.6 finally justify the terminology: it is R which not only is R-complete, but, e.g., polynomials are (B-dense in C and therefore) R-dense in R.

3. FUNCTIONALS ON R

Let $R' = R'[a, b]$ be the set of sublinear functionals F on $R = R[a, b]$, i.e.,

$$|F(f+g)| \leqslant |Ff| + |Fg|, \qquad |F(\alpha f)| = |\alpha|\, |Ff|, \tag{3.1}$$

for $f, g \in R$, $\alpha \in \mathbb{C}$. Concerning continuity, recall that R is indeed endowed with two concepts of convergence: uniform convergence (i.e., in B-norm) and R-convergence. Accordingly, two classes of continuous functionals are distinguished: $R^\wedge = R^\wedge[a, b]$, the subclass of functionals, B-continuous at the origin ($|F|$ is B-continuous on R) or, equivalently, bounded on R,

$$R^\wedge := \{F \in R' : \|F\| = \|F\|_{R^\wedge} := \sup\{|Ff| : f \in R, \|f\|_B \leqslant 1\} < \infty\}, \quad (3.2)$$

and $R^* = R^*[a, b]$, the subclass of functionals, (sequentially) R-continuous at the origin, i.e.,

$$R^* := \{F \in R' : f_n \in R, R\text{-}\lim_{n \to \infty} f_n = 0 \Rightarrow \lim_{n \to \infty} Ff_n = 0\}. \quad (3.3)$$

Note that, if $F \in R^*$ is linear, then F is R-continuous at each $f \in R$, thus for any sequence $\{f_n\} \subset R$ the implication

$$R\text{-}\lim_{n \to \infty} f_n = f \Rightarrow \lim_{n \to \infty} Ff_n = Ff$$

holds true. Since uniform convergence implies R-convergence, each $F \in R^*$ is bounded, thus $R^* \subset R^\wedge \subset R'$.

With $F \in R^\wedge$ let us associate the functional

$$F^*(I) := \sup\{|F(f\chi_I)| : f \in R, \|f\| \leqslant 1\} \qquad (I \in V). \quad (3.4)$$

Obviously, $F^*([a, b]) = \|F\|$.

Lemma 3.1. *Let $F \in R^\wedge$. Then F^* is monotone and subadditive on V, i.e., for $I, J \in V$,*

$$F^*(I) \leqslant F^*(J) \qquad (I \subset J), \quad (3.5)$$

$$F^*(I \cup J) \leqslant F^*(I) + F^*(J). \quad (3.6)$$

Proof. Let $f \in R$ with $\|f\| \leqslant 1$. If $I \subset J$, then (3.5) follows in view of

$$|F(f\chi_I)| = |F((f\chi_I)\,\chi_J)| \leqslant F^*(J)$$

since $\chi_I = \chi_I \chi_J$ and $\|f\chi_I\| \leqslant 1$. On the other hand, $\chi_{I \cup J} = \chi_I + \chi_{J \setminus I}$ for arbitrary $I, J \in V$ so that

$$F^*(I \cup J) \leqslant F^*(I) + F^*(J \setminus I) \leqslant F^*(I) + F^*(J)$$

since F is subadditive. ∎

PROPOSITION 3.2. *$F \in R'$ belongs to R^* if and only if $F \in R^\wedge$ and*

$$F^*(I_n) = o(1) \qquad (I_n \downarrow 0). \tag{3.7}$$

Moreover, for $F \in R^\wedge$ the condition

$$|Ff| \leqslant K \int |f| \qquad (f \in R) \tag{3.8}$$

is sufficient for $F \in R^*$.

Proof. Let $F \in R^\wedge$ satisfy (3.7) and assume $\{f_n\} \subset R$ be such that R-$\lim f_n = 0$. Then $\|f_n\| \leqslant M$ by (2.1) and by Proposition 2.2(iv) for any $\varepsilon > 0$ there exists $I_n \downarrow 0$ with $|f_n(x)| \leqslant \varepsilon$ on $\complement I_n$. This implies

$$|Ff_n| \leqslant |F(f_n \chi_{I_n})| + |F(f_n \chi_{\complement I_n})| \leqslant MF^*(I_n) + \|F\| \varepsilon,$$

thus $Ff_n = o(1)$. To establish the converse, assume that (3.7) is violated, i.e., $F^*(I_n) \geqslant 2\varepsilon_0 > 0$ for some $I_n \downarrow 0$ and all $n \in \mathbb{N}$ (cf. (3.5)). By Definition (3.4) there are $f_n \in R$ with $\|f_n\| \leqslant 1$ and $|F(f_n \chi_{I_n})| \geqslant \varepsilon_0$. But $g_n := f_n \chi_{I_n}$ is R-convergent to zero since

$$\|g_n\| \leqslant 1, \qquad \overline{\int} \sup_{k \geqslant n} |g_k| \leqslant \overline{\int} \sup_{k \geqslant n} |f_k| \, \chi_{I_n} \leqslant \mu(I_n) \tag{3.9}$$

which contradicts $F \in R^*$. Concerning the supplement, note that (3.8) implies

$$F^*(I) \leqslant \sup_{\|f\| \leqslant 1} K \int |f \chi_I| = K\mu(I). \quad \blacksquare$$

For example, the Riemann integral $Qf := \int f$ is R-continuous by (3.8) (or, since $Q^*(I) = \mu(I)$). On the other hand, the point evaluation functional at $x_0 \in [a, b]$,

$$F_{x_0} f := f(x_0), \qquad (f \in R) \tag{3.10}$$

is certainly bounded (and B-continuous), but F_{x_0} is not R-continuous. Indeed, for $f_n = \chi_{\{x_0\}}$ one obviously has $f_n \in R$ with R-$\lim f_n = 0$ but $F_{x_0} f_n = 1$.

A sequence $\{F_n\} \subset R'$ is called equi-R-continuous if

$$f_n \in R, \qquad R\text{-}\lim_{n \to \infty} f_n = 0 \Rightarrow \sup_{k \in \mathbb{N}} |F_k f_n| = o(1). \tag{3.11}$$

It follows that indeed $\sup_k |F_k| \in R^*$. A sequence $\{F_n\} \subset R'$ is said to be continuously R-convergent (at the origin) if (cf. [8, p. 197])

$$f_n \in R, \qquad R\text{-}\lim_{n \to \infty} f_n = 0 \Rightarrow F_n f_n = o(1). \tag{3.12}$$

PROPOSITION 3.3. *A sequence $\{F_n\} \subset R'$ is equi-R-continuous if and only if $F_n \in R^*$ for each $n \in \mathbb{N}$ and $\{F_n\}$ is continuously R-convergent. Moreover, continuous R-convergence of $\{F_n\} \subset R^\wedge$ necessarily implies equi-B-continuity (at the origin), thus the uniform boundedness,*

$$\|F_n\| \leqslant M \qquad (n \in \mathbb{N}). \tag{3.13}$$

Proof. Obviously, (3.11) implies (3.12). Conversely, assume that (3.11) does not hold true so that there exists $f_n \in R$, R-$\lim f_n = 0$ such that for some $\varepsilon_0 > 0$ (at least for a subsequence),

$$\sup_{k \in \mathbb{N}} |F_k f_n| \geqslant 2\varepsilon_0 > 0 \qquad (n \in \mathbb{N}).$$

This implies $|F_{k_n} f_n| \geqslant \varepsilon_0$ for some $k_n \in \mathbb{N}$. The subsequence $\{k_n\}$ is unbounded since, if $k_n \leqslant K$, then

$$\varepsilon_0 \leqslant |F_{k_n} f_n| \leqslant F f_n := \max_{1 \leqslant k \leqslant K} |F_k f_n|, \tag{3.14}$$

but $Ff_n = o(1)$ in view of $F \in R^*$, a contradiction. Hence there is a subsequence $\{k_{n_j}\}$, strictly increasing to infinity. Since each subsequence of $\{F_n\}$ is continuously R-convergent too, one has $F_{k_{n_j}} f_{n_j} = o(1)$, in contrast to (3.14). Concerning (3.13) suppose that $\|F_n\| \to \infty$. Then there exists $f_n \in R$ with $\|f_n\| \leqslant 1$ and $|F_n f_n| \to \infty$. Since $g_n := f_n / |F_n f_n|^{1/2}$ is R-convergent to zero, in fact even uniformly, and since $|F_n g_n| = |F_n f_n|^{1/2} \to \infty$, this violates (3.12). ∎

PROPOSITION 3.4. *A sequence $\{F_n\} \subset R'$ is, equi-R-continuous if and only if it satisfies* (3.13) *and*

$$\sup_{k \in \mathbb{N}} F_k^*(I_n) = o(1) \qquad (I_n \downarrow 0). \tag{3.15}$$

Moreover, a sufficient criterion for (3.11) *is given by* (3.13) *together with* (*cf.* (3.8))

$$|F_n f| \leqslant K \int |f| \qquad (f \in R). \tag{3.16}$$

Proof. The assertions are an easy consequence of Proposition 3.2 as applied to $F := \sup_k |F_k|$ since $F^*(I) = \sup_k F_k^*(I)$. ∎

PROPOSITION 3.5. *A sequence* $\{F_n\} \subset R^\wedge$ *is continuously R-convergent if and only if it satisfies* (3.13) *and one of the following properties*:

$$F_n^*(I_n) = o(1) \qquad (I_n \downarrow 0), \tag{3.17}$$

$$\limsup_{k \to \infty} F_k^*(I_n) = o(1) \qquad (I_n \downarrow 0). \tag{3.18}$$

Moreover, (3.12) *for* $\{F_n\} \subset R^\wedge$ *necessarily implies that the functional*

$$F := \limsup_{k \to \infty} |F_k| \tag{3.19}$$

belongs to R^*.

Proof. Let $\{F_n\} \subset R^\wedge$ satisfy (3.12). Then (3.13) is necessary by Proposition 3.3. Assume that (3.18) is violated. Then there exists $I_n \downarrow 0$ such that

$$\limsup_{k \to \infty} F_k^*(I_n) \geqslant 2\varepsilon_0 > 0,$$

for infinitely many $n \in \mathbb{N}$, thus for all $n \in \mathbb{N}$ by Lemma 3.1. This implies that there are a subsequence F_{k_n} and elements $f_n \in R$ with $\|f_n\| \leqslant 1$ and $|F_{k_n}(f_n \chi_{I_n})| \geqslant \varepsilon_0$. Since $g_n := f_n \chi_{I_n}$ is R-convergent to zero (cf. (3.9)), the subsequence $\{F_{k_n}\}$, and hence $\{F_n\}$ too, is not continuously R-convergent, a contradiction. Now (3.18) implies (3.17) since for a given sequence $I_n \downarrow 0$ one has by Lemma 3.1

$$F_k^*(I_k) \leqslant F_k^*(I_n) \qquad (k \geqslant n),$$

$$\limsup_{k \to \infty} F_k^*(I_k) \leqslant \limsup_{k \to \infty} F_k^*(I_n) = o(1) \qquad (n \to \infty).$$

That (3.13), (3.17) are sufficient for (3.12) follows analogously from the first part of the proof of Proposition 3.2 (take $F = F_n$) so that the equivalence assertion is established. Concerning the R-continuity of F (cf. (3.19)), by Proposition 3.2 it is enough to observe that $F \in R^\wedge$ by (3.13) and that $F^*(I) \leqslant \limsup F_k^*(I)$. ∎

As a typical example let us consider the quadrature (cubature) formula

$$Q_m f := \sum_{k=1}^{m} a_{km} f(x_{km}) \qquad (a_{km} \in \mathbb{C},\ x_{km} \in [a, b]) \tag{3.20}$$

which is certainly bounded but not R-continuous (cf. (3.10)). Since

$$Q_m^*(I) = \sum_{x_{km} \in I} |a_{km}| \qquad (I \in V), \tag{3.21}$$

Condition (3.18) corresponds to the original one of Pólya [12] on the semicontinuity of the associated "Intervallvereinfunktion"

$$\Lambda(I) := \limsup_{m \to \infty} Q_m^*(I) \qquad (I \in V). \tag{3.22}$$

4. Theorems of Banach–Steinhaus-Type

Let us start with the classical Banach–Steinhaus theorem as applied to the Banach space R under B-norm.

Proposition 4.1. *For* $\{F_n\} \subset R^\wedge$ *let* $U \subset R$ *be B-dense in R. Then*

$$F_n f = o(1) \qquad (f \in R) \tag{4.1}$$

is equivalent to (3.13) *and*

$$F_n g = o(1) \qquad (g \in U). \tag{4.2}$$

Of course, the disadvantage of this statement is the lack of suitable B-dense subsets. In fact, in view of Theorem 2.6, Corollary 2.7 one should weaken B-dense to R-dense. In this connection a first contribution to the characterization of convergence on R reads as follows.

Proposition 4.2. *For* $\{F_n\} \subset R^\wedge$ *let* $U \subset R$ *be R-dense in R. Then* (4.1) *is equivalent to* (4.2), (3.13), *and* $F := \limsup |F_k| \in R^*$.

Proof. Of course, (4.1) implies (4.2), (3.13) (cf. Proposition 4.1) as well as the R-continuity of F (well-defined by (3.13)) since $F = 0$. Conversely, the R-continuous F vanishes on the R-dense subset U, thus $F = 0$ on the whole space. ∎

Returning to Proposition 4.1, obviously the boundedness of each F_n is equivalent to the B-continuity of F_n, and (3.13) coincides with the equi-B-continuity of $\{F_n\}$ (at the origin). It is therefore natural to expect that a weakening to R-dense subsets corresponds to a strengthening of B-continuity to R-continuity. In fact,

Theorem 4.3. *For* $\{F_n\} \subset R^*$ *let* $U \subset R$ *be R-dense in R. Then* (4.1) *is equivalent to* (4.2) *together either with the equi-R-continuity or with the continuous R-convergence of* $\{F_n\}$.

Proof. For the sufficiency see Propositions 4.2, 3.3, 3.5. Concerning the necessity let us apply the gliding hump method (see also [13, 14] for a

Baire approach in the framework of Fréchet spaces). First of all, (4.1) implies (4.2) trivially and (3.13) by Proposition 4.2. Assume then that $\{F_n\}$ is not equi-R-continuous, thus not continuously R-convergent by Proposition 3.3. In view of (3.17) there exists $I_n \downarrow 0$ such that

$$F_n^*(I_n) \geqslant \varepsilon_0 > 0 \qquad (n \in \mathbb{N}) \tag{4.3}$$

(at least for a subsequence). Since $F_n \in R^*$, one also has (cf. (3.7))

$$\lim_{k \to \infty} F_n^*(I_k) = 0. \tag{4.4}$$

Therefore one may successively construct a subsequence $\{n_k\} \subset \mathbb{N}$ with $n_1 = 1$ such that

$$F_{nk}^*(I_{nk+1}) \leqslant \varepsilon_0/8. \tag{4.5}$$

In view of Lemma 3.1 and (4.3) this implies that for $J_k := I_{n_k} \backslash I_{n_{k+1}}$,

$$F_{nk}^*(J_k) \geqslant F_{nk}^*(I_{nk}) - F_{nk}^*(I_{nk+1}) \geqslant 7\varepsilon_0/8 \tag{4.6}$$

so that there exists $f_k \in R$, $\|f_k\| \leqslant 1$ such that for $g_k := f_k \chi_{J_k}$,

$$|F_{nk} g_k| \geqslant \varepsilon_0/2. \tag{4.7}$$

In view of (4.1) and $g_k \in R$ one may again select a subsequence $\{k_p\} \subset \mathbb{N}$ such that

$$|F_{n_{k_p}} h_{p-1}| \leqslant \varepsilon_0/8, \qquad h_{p-1} := \sum_{j=1}^{p-1} g_{k_j}. \tag{4.8}$$

Now the functions g_k have disjoint supports $(\subset J_k)$ so that $\{h_p\}$ is equibounded by 1 (cf. (2.1)) with $(q > p \geqslant m)$

$$|h_q(x) - h_p(x)| = \left| \sum_{j=p+1}^{q} g_{k_j}(x) \right| \chi_{I_{n_{k_{p+1}}}}(x) \leqslant \chi_{I_{n_{k_m}}}(x),$$

$$\bar{\int} \sup_{q > p \geqslant m} |h_q - h_p| \leqslant \mu(I_{n_{k_m}}) = o(1) \qquad (m \to \infty).$$

Thus (2.9) is fulfilled and the proof of Proposition 2.4 and Theorem 2.5 yields that the pointwise limit (cf. (2.10))

$$f(x) := \limsup_{p \to \infty} \sum_{j=1}^{p} g_{k_j}(x) = \lim_{p \to \infty} \sum_{j=1}^{p} g_{k_j}(x)$$

belongs to R. Now one obtains

$$f-h_p=(f-h_p)\,\chi_{I_{nk_{p+1}}},$$
$$|F_{nk_p}(f-h_p)|\leqslant F^*_{nk_p}(I_{nk_{p+1}})\leqslant F^*_{nk_p}(I_{nk_p+1})\leqslant \varepsilon_0/8$$

by Lemma 3.1 and (4.5) so that by (4.7), (4.8),

$$|F_{nk_p}f|\geqslant |F_{nk_p}\,g_{k_p}|-|F_{nk_p}h_{p-1}|-|F_{nk_p}(f-h_p)|\geqslant \varepsilon_0/4,$$

a contradiction to (4.1). ∎

Let us mention that the present general equivalence theorem (together with Proposition 3.4) should be compared with results of [7] (see also [5] for related material) which are concerned with functionals of the special type $\int f\varphi_n$ for integrable φ_n.

Since quadrature rules are not R-continuous, Theorem 4.3 does not yet cover Pólya's original result. But for this kind of functional R-continuity may indeed be dropped.

THEOREM 4.4. *Let $\{Q_m\}$ be a sequence of quadrature formulas (cf. (3.20)). Then*

$$\lim_{m\to\infty} Q_m f=Qf:=\int f \qquad (f\in R) \tag{4.9}$$

if and only if

$$\lim_{m\to\infty} Q_m\, g=Qg \qquad (g\in C) \tag{4.10}$$

and $\{Q_m\}$ is continuously R-convergent.

Proof. Essentially one may proceed as for Theorem 4.3 so that we only indicate how to derive analogues of (4.3), (4.4) in connection with the necessity. To this end, set $\Delta_m:=\{x_{km}: 1\leqslant k\leqslant m\}$ so that

$$\lim_{m\to\infty} Q^*_m(\Delta_j)=0 \qquad (j\in\mathbb{N}) \tag{4.11}$$

by (3.6) and (cf. (4.9))

$$\lim_{m\to\infty} Q^*_m(\{x\})=\lim_{m\to\infty} |Q_m\chi_{\{x\}}|=0 \qquad (x\in[a,b]).$$

Again assume that $\{Q_m\}$ is not continuously R-convergent, i.e., there exists $I_m\downarrow 0$ such that

$$Q^*_m(I_m)\geqslant 2\varepsilon_0>0 \qquad (m\in\mathbb{N}). \tag{4.12}$$

Then one may successively construct a subsequence $\{m_n\} \subset \mathbb{N}$ such that (cf. (3.6), (4.11))

$$Q^*_{m_n}(X_{n-1}) + \mu(I_{m_n}) \leqslant \varepsilon_0, \qquad X_{n-1} := \bigcup_{k=1}^{n-1} \Delta_{m_k}.$$

Now consider $\tilde{I}_n := I_{m_n} \setminus X_{n-1} \downarrow 0$ and $F_n := |Q_{m_n} - Q|$. Then

$$F^*_n(\tilde{I}_n) \geqslant Q^*_{m_n}(I_{m_n}) - Q^*_{m_n}(X_{n-1}) - Q^*(I_{m_n}) \geqslant \varepsilon_0.$$

In view of (3.21) and $\tilde{I}_k \subset \complement\, \Delta_{m_n}$ for $k \geqslant n+1$ one has $Q^*_{m_n}(\tilde{I}_k) = 0$ so that

$$\lim_{k \to \infty} F^*_n(\tilde{I}_k) = \lim_{k \to \infty} Q^*(\tilde{I}_k) = 0.$$

Hence (4.3), (4.4) are fulfilled. ▮

The standard situation in applications is that well known classical results establish convergence for polynomials or continuous functions. Theorems 4.3, 4.4 then extend those results to all Riemann integrable functions, provided continuous *R*-convergence can be established. Let us illustrate the latter aspect in connection with the convergence of quadrature (cubature) rules.

First, consider the compound formula ($f \in R[0, \bar{1}]$)

$$Q_m f = m^{-N} \sum_{i \in \Gamma_{m-1}} \sum_{k=1}^{s} b_k f\left(\frac{i + x_k}{m}\right) \tag{4.13}$$

with $b_k \in \mathbb{C}$, $x_k \in [0, \bar{1}]$ (see proof of Theorem 2.6 for the notations). It is well known that $Q_m f$ converges to Qf on C if and only if the weights satisfy (cf. [3, p. 21]; by the way, the elementary argument there immediately extends to R)

$$\sum_{k=1}^{s} b_k = 1. \tag{4.14}$$

Corollary 4.5. *The compound quadrature procedure* (4.13) *satisfies* (4.9) *if and only if* (4.14) *is valid.*

Proof. By Theorem 4.4 it is enough to check that $\{Q_m\}$ is continuously *R*-convergent, thus to show (3.13), (3.18). But this is an immediate consequence of (cf. (3.22))

$$\Lambda(I) = \lim_{m \to \infty} Q^*_m(I) = B\mu(I) \qquad (I \in V) \tag{4.15}$$

with $B := \sum_{k=1}^{s} |b_k|$. Indeed, setting

$$J_m^1 := \bigcup \{S_{im} : S_{im} \subset I\}, \qquad J_m^2 := \bigcup \{S_{im} : S_{im} \cap I \neq \varnothing\}$$

one obtains $J_m^1 \subset I \subset J_m^2$ and (cf. (3.21))

$$Q_m^*(J_m^1) = m^{-N} \sum_{S_{im} \subset I} \sum_{k=1}^{s} |b_k| = B\mu(J_m^1),$$

$$Q_m^*(J_m^2) = m^{-N} \sum_{S_{im} \cap I \neq \varnothing} \sum_{k=1}^{s} |b_k| = B\mu(J_m^2).$$

But $\lim_{m \to \infty} \mu(J_m^k) = \mu(I)$ so that (4.15) follows in view of (3.5). ∎

Next consider positive quadrature formulas, i.e., $a_{km} > 0$ in (3.20). It was shown in [6] that such a process converges on $R[-1, 1]$ if it is additionally interpolatory (so that trivially $\lim_{n \to \infty} Q_n x^k = Qx^k$, $k \in \mathbb{P}$). This result is now regained by

COROLLARY 4.6. *If Q_m is a positive quadrature formula, then $\{Q_m\}$ satisfies* (4.9) *if and only if*

$$\lim_{m \to \infty} Q_m(x_1^{j_1} \cdots x_N^{j_N}) = Q(x_1^{j_1} \cdots x_N^{j_N}),$$

for any multiindex $(j_1, ..., j_N) \in \mathbb{P}^N$.

Proof. Since the procedure $\{Q_m\}$ is positive and converges for polynomials, it converges on C (cf. [3, p. 35]). To apply Theorem 4.4 let us show that $\{Q_m\}$ is continuously R-convergent (and thus $\{Q_m - Q\}$, too). Now $Q_m^*(I) = Q_m \chi_I$ so that (3.13), (3.18) follow if

$$(\Lambda(I) =) \lim_{m \to \infty} Q_m \chi_I = Q\chi_I \qquad (= \mu(I)), \tag{4.16}$$

for any $I \in V$, or even only for any subinterval $I := [c, d] \subset [a, b]$, since Q_m, Q are linear. To this end, first note that there exist $h_n^k \in C$ with

$$R\text{-}\lim_{n \to \infty} h_n^k = \chi_I, \qquad h_n^1 \leqslant \chi_I \leqslant h_n^2.$$

Indeed, take (cf. [6])

$$h_n^k(x) := \prod_{j=1}^{N} h_{jn}^k(x_j)$$

with the trapezoidal functions ($t \in \mathbb{R}$)

$$h_{jn}^1(t) := \begin{cases} 1, & t \in [c_j + 1/n, d_j - 1/n] \\ 0, & t \notin [c_j, d_j] \\ \text{linear}, & \text{else}, \end{cases}$$

$$h_{jn}^2(t) := \begin{cases} 1, & t \in [c_j, d_j] \\ 0, & t \notin [c_j - 1/n, d_j + 1/n] \\ \text{linear}, & \text{else}. \end{cases}$$

Since Q_m is positive and Q is R-continuous, one obtains

$$Q_m h_n^1 \leqslant Q_m \chi_I \leqslant Q_m h_n^2,$$
$$Q\chi_I = \lim_{n\to\infty} Qh_n^k = \lim_{n\to\infty} \lim_{m\to\infty} Q_m h_n^k \qquad (k = 1, 2);$$

thus (4.16) follows. ∎

5. R-Convergence of Operators

In this section we establish sufficient conditions for the R-convergence of a sequence of operators and discuss some significant examples from approximation theory. To this end, an operator T of R into itself is called bounded if (cf. (3.2))

$$\|T\| := \sup\{\|Tf\|_B : f \in R, \|f\|_B \leqslant 1\}$$

is finite. Then the operator ($x \in [a, b]$, cf. (3.4))

$$T^*(I)(x) := \sup\{|T(f\chi_I)(x)| : f \in R, \|f\| \leqslant 1\} \tag{5.1}$$

is well-defined not only for $I \in V$ but also for each Riemann measurable subset, in particular for

$$K_{\delta,x} := \{y \in [a, b] : |y - x| \geqslant \delta\} \qquad (\delta > 0, x \in [a, b]). \tag{5.2}$$

In these terms one has

Theorem 5.1. *Let $\{T_n\}$ be a sequence of linear operators of R into itself. Then the conditions*

$$\|T_n\| = \mathcal{O}(1), \tag{5.3}$$

$$T_n 1 = 1, \tag{5.4}$$

$$R\text{-}\lim_{n\to\infty} T_n^*(K_{\delta,x})(x) = 0 \qquad (\delta > 0) \tag{5.5}$$

are sufficient for

$$R\text{-}\lim_{n\to\infty} T_n f = f \qquad (f \in R), \tag{5.6}$$

thus for $\{T_n\}$ *to be a linear R-approximation process on R.*

Proof. Consider the sublinear functionals

$$F_n f := \overline{\int} \sup_{k \geqslant n} |T_k f - f|, \qquad Ff := \limsup_{n\to\infty} F_n f.$$

By (5.3) these functionals are well-defined on R and (5.6) coincides with the statement that $F=0$ on R. Let $g \in C$ with (first) modulus of continuity

$$\omega(g, \delta) := \sup\{|g(u) - g(x)| : u, x \in [a, b], |u - x| \leqslant \delta\}.$$

For $x \in [a, b]$, $\delta > 0$ one obtains by (5.1)–(5.4) (with $\|T_k\| \leqslant M$),

$$\begin{aligned} |T_k g(x) - g(x)| &= |T_k[g - g(x)](x)| \\ &\leqslant |T_k[(g - g(x))\,\chi_{K_{\delta,x}}](x)| + \|T_k\|\;\|[g - g(x)]\,\chi_{\complement K_{\delta,x}}\| \\ &\leqslant 2\,\|g\|\;T_k^*(K_{\delta,x})(x) + M\omega(g, \delta). \end{aligned}$$

By the peaking property (5.5) this implies

$$Fg \leqslant M\mu([a, b]) \lim_{\delta\to 0+} \omega(g, \delta) = 0 \qquad (g \in C). \tag{5.7}$$

Now let $I \in V$, $\delta > 0$. With

$$I_\delta := \{x \in [a, b]\text{: there exists } y \in I \text{ with } |x_j - y_j| \leqslant \delta,\ 1 \leqslant j \leqslant N\}$$

one has $I_\delta \in V$ and $\mu(I_\delta) \to \mu(I)$ for $\delta \to 0+$. Moreover, $I \subset K_{\delta,x}$ for each $x \in \complement I_\delta$ so that (cf. (3.5))

$$T_k^*(I)(x) \leqslant T_k^*(K_{\delta,x})(x) \qquad (x \in \complement I_\delta).$$

But this yields

$$\begin{aligned} F_n^*(I) &\leqslant \overline{\int} \sup_{k \geqslant n} T_k^*(I) + \mu(I) \leqslant \overline{\int} \sup_{k \geqslant n} T_k^*(K_{\delta,x})(x)\,dx + M\mu(I_\delta) + \mu(I), \\ F^*(I) &\leqslant \lim_{\delta\to 0+} M\mu(I_\delta) + \mu(I) = (M+1)\,\mu(I) \end{aligned}$$

so that F is R-continuous by Proposition 3.2. Therefore (5.7) and Theorem 2.6 yield the assertion. ∎

This theorem is indeed closely related to the classical approximate identity argument (peaking property) of Fourier analysis and approximation theory. To this end, let $R_{2\pi}$ be the set of functions 2π-periodic in each variable and Riemann integrable over $[-\pi, \pi]^N$. For $\{k_n\} \subset R_{2\pi}$ consider the convolution operators

$$T_n f(x) = k_n * f(x) := \int_{[-\pi,\pi]^N} k_n(x-u) f(u)\, du \qquad (f \in R_{2\pi}). \tag{5.8}$$

$\{k_n\}$ is called an approximate identity (in the classical sense, cf. [4, p. 30]) if

$$\int |k_n| = \mathcal{O}(1) \qquad (n \to \infty), \tag{5.9}$$

$$\int k_n = 1 \qquad (n \in \mathbb{N}), \tag{5.10}$$

$$\lim_{n \to \infty} \int_{K_{\delta,0}} |k_n| = 0 \qquad (\delta > 0). \tag{5.11}$$

Obviously, (5.9), (5.10) coincide with (5.3), (5.4), and (5.11) is equivalent to (5.5) since

$$T_n^*(K_{\delta,x})(x) = \int_{K_{\delta,0}} |k_n| \qquad (x \in [-\pi, \pi]^N).$$

Hence Theorem 5.1 delivers (without any additional assumptions on k_n)

THEOREM 5.2. *Let* $\{k_n\} \subset R_{2\pi}$ *satisfy* (5.9)–(5.11). *Then the convolution operators* (5.8) *establish an R-approximation process on* $R_{2\pi}$.

Note that $F_n f := \int |k_n * f - f|$ is equi-R-continuous in view of (5.9) and (3.16).

It is important to observe that Theorem 5.1 does not assume any R-continuity of the operators so that applications are possible, even if point evaluation functionals are involved. For example, consider the Bernstein polynomials for $f \in R[0, 1]$,

$$B_n f := \sum_{k=0}^{n} f(k/n)\, p_{kn}, \qquad p_{kn}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

COROLLARY 5.3. *The Bernstein polynomials constitute an R-approximation process on* $R[0, 1]$.

Proof. Again the sufficient conditions of Theorem 5.1 are fulfilled since in particular (cf. [9, p. 6]),

$$B_n^*(I)(x) = \sum_{k/n \in I} p_{kn}(x), \qquad \sum_{|k/n - x| \geq \delta} p_{kn}(x) \leq 1/4n\delta^2 = o(1) \qquad (n \to \infty). \quad \blacksquare$$

References

1. A. ALEXIEWICZ, On sequences of operations I, II, *Studia Math.* **11** (1950), 1–30, 200–236.
2. T. M. APOSTOL, "Mathematical Analysis," Addison–Wesley, Reading, MA, 1971.
3. H. PRASS, "Quadraturverfahren," Vandenhoeck & Ruprecht, Göttingen/Zürich, 1977.
4. P. L. BUTZER AND R. J. NESSEL, "Fourier Analysis and Approximation," Vol. I, Birkhäuser, Basel, 1971; Academic Press, New York, 1971.
5. J. DIEUDONNÉ, Sur la convergence des suites de mesures de Radon, *An. Acad. Brasil. Ciênc.* **23** (1951), 21–38, 277–282.
6. L. FEJÉR, Mechanische Quadraturen mit positiven Cotesschen Zahlen, *Math. Z.* **37** (1933), 287–309.
7. G. FICHTENHOLZ, Sur les suites convergentes des intégrales définies, *Anz. Krakau Akad. Wiss. Math.-Naturw. Kl. A* (1923), 91–117.
8. K. KURATOWSKI, "Topology," Vol. I, Academic Press, New York, 1966.
9. G. G. LORENTZ, "Bernstein Polynomials," Univ. Toronto Press, Toronto, 1953.

10a. H. MEVISSEN, R. J. NESSEL, AND E. VAN WICKEREN, On the Riemann convergence of positive linear operators, *in* "Proc. Constructive Function Theory 1986, Edmonton," *Rocky Mountain J. Math.*, in press.

10b. H. MEVISSEN, R. J. NESSEL, AND E. VAN WICKEREN, On the sharpness of error bounds for compound quadrature rules in the space of Riemann integrable functions, *in* "Constructive Theory of Functions 1987; Proc. Conf. Varna, 1987," in press.

10c. H. MEVISSEN, R. J. NESSEL, AND E. VAN WICKEREN, A quantitative Bohman–Korovkin theorem and its sharpness for Riemann integrable functions, *Z. Anal. Anwendungen* **7** (1988), in press.

11. W. ORLICZ, Linear operations in Saks spaces I, *Studia Math.* **11** (1950), 237–272.
12. G. PÓLYA, Über die Konvergenz von Quadraturverfahren, *Math. Z.* **37** (1933), 264–286.
13. E. VAN WICKEREN, "Resonanzprinzipien der Approximationstheorie aus Baire'scher Sicht," Dissertation, RWTH Aachen, 1984.
14. E. VAN WICKEREN, A Baire approach to quantitative resonance principles, *Numer. Funct. Anal. Optim.* **9** (1987), 147–180.
15. E. VAN WICKEREN, Sequential convergence in the space of functions Riemann integrable on arbitrary sets, *in* "Function Spaces; Proc. Conf. Poznań, 1986" (J. Musielak, Ed.), pp. 85–92, Teubner, Leipzig, 1988.
16. E. VAN WICKEREN, On the approximation of Riemann integrable functions by Bernstein polynomials, *in* "Proc. Conf. Havanna, 1987," in press.
17. E. VAN WICKEREN, Direct and inverse theorems for Bernstein polynomials in the space of Riemann integrable functions, *Constr. Approx.*, in press.